

MAS212 Linear Algebra I 2007-2008

Coursework 3 — Solutions

Exercise 1.

- (a) Suppose that A and C are square matrices with $AC = I$. In order to prove the assertions, we only need to show that C is invertible, since then

$$A = AI = ACC^{-1} = IC^{-1} = C^{-1},$$

so $CA = CC^{-1} = I$. Now, by the Invertible Matrix Theorem, in order to prove that C is invertible, it suffices to show that $C\mathbf{x} = \mathbf{0}$ has only the trivial solution. But if $C\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = I\mathbf{x} = AC\mathbf{x} = A\mathbf{0} = \mathbf{0},$$

so the only solution of $C\mathbf{x} = \mathbf{0}$ is the trivial solution.

- (b) There are many examples (why?); the simplest is perhaps the following one: let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{but} \quad CA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In other words, the result in (a) is false if A and C are not square.

Exercise* 2.

- (a) Using Gauss-Jordan inversion we find

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array} \right) \\ & \xrightarrow{R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array} \right) \xrightarrow{\substack{R_1 - 3R_3 \\ R_2 + R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -6 & -3 \\ 0 & 1 & 0 & -3 & 3 & 1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array} \right). \end{aligned}$$

Thus A is invertible and

$$A^{-1} = \begin{pmatrix} 7 & -6 & -3 \\ -3 & 3 & 1 \\ -2 & 2 & 1 \end{pmatrix}.$$

- (b) The system can be written in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & -2 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix}.$$

Thus, by (a),

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 7 & -6 & -3 \\ -3 & 3 & 1 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

that is, the solution set of the system is $\{(2, -3, 1)\}$.

Exercise* 3.

(a) Using Gauss-Jordan inversion we find

$$\begin{aligned} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 5 & 1 & -20 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array} \right) & \xrightarrow{R_2 - 5R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array} \right) \\ & \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right) \xrightarrow{R_1 + 4R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right). \end{aligned}$$

(b) By (a), the elementary matrices corresponding to the elementary row operations used to bring A to the identity are

$$\begin{aligned} E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{take } I_3 \text{ and subtract 5 times row 1 from row 2}); \\ E_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad (\text{take } I_3 \text{ and multiply row 3 by } \frac{1}{2}); \\ E_3 &= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{take } I_3 \text{ and add 4 times row 3 to row 1}), \end{aligned}$$

with

$$E_3 E_2 E_1 A = I.$$

Thus the inverse of A can be written as

$$A^{-1} = E_3 E_2 E_1,$$

while

$$A = (A^{-1})^{-1} = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}.$$

The inverse of the elementary matrix E_l ($l = 1, 2, 3$) above is given by performing that row operation on I_3 that undoes the effect of the row operation used to obtain E_l from I_3 , that is,

$$\begin{aligned} E_1^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{take } I_3 \text{ and add 5 times row 1 to row 2}); \\ E_2^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (\text{take } I_3 \text{ and multiply row 3 by 2}); \\ E_3^{-1} &= \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{take } I_3 \text{ and subtract 4 times row 3 from row 1}). \end{aligned}$$

Exercise* 4. The cofactor expansion across the second row gives

$$\begin{vmatrix} 1 & 0 & -1 \\ -2 & 7 & 3 \\ -3 & 5 & 4 \end{vmatrix} = -(-2) \begin{vmatrix} 0 & -1 \\ 5 & 4 \end{vmatrix} + 7 \begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} = -(-2) \cdot 5 + 7 \cdot 1 - 3 \cdot 5 = 2,$$

while the cofactor expansion down the third column gives

$$\begin{vmatrix} 1 & 0 & -1 \\ -2 & 7 & 3 \\ -3 & 5 & 4 \end{vmatrix} = (-1) \begin{vmatrix} -2 & 7 \\ -3 & 5 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} + 4 \begin{vmatrix} 1 & 0 \\ -2 & 7 \end{vmatrix} = (-1) \cdot 11 - 3 \cdot 5 + 4 \cdot 7 = 2,$$

as well.

Exercise* 5. Remember that adding multiples of one row to another row does not change the determinant, while swapping two rows changes the sign of the determinant.

- (a) Perhaps the quickest way to evaluate this determinant is to spot that row 2 and row 4 only differ by one entry. So we first subtract row 2 from row 4, then we expand across row 4, and finally we expand across row 1:

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 3 & 4 & 1 & 2 \\ 5 & 3 & 1 & 2 \\ 3 & 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 3 & 4 & 1 & 2 \\ 5 & 3 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 1 \\ 5 & 3 & 1 \end{vmatrix} = 3 \left[-1 \begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} \right]$$

$$= 3[(-1) \cdot (-2) + 2 \cdot (-11)] = -60.$$

- (b) Here we repeatedly use row operations of type III to bring the matrix to an upper triangular one, and then use the fact that the determinant of a triangular matrix is the product of the diagonal entries:

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ -1 & 0 & -2 & -5 \\ 3 & 4 & -1 & 15 \\ -2 & 0 & -1 & -20 \end{vmatrix} \xrightarrow{\substack{R_2 + R_1 \\ R_3 - 3R_1 \\ R_4 + 2R_1}} \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 4 & -3 & -14 \end{vmatrix} \xrightarrow{\substack{R_3 + R_2 \\ R_4 - 2R_2}} \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 3 & -10 \end{vmatrix}$$

$$\xrightarrow{R_4 + 3R_3} \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1 \cdot 2 \cdot (-1) \cdot 2 = -4.$$

- (c) The easiest way to evaluate this determinant is to first swap rows 1 and 5, and then to swap rows 4 and 5, which brings the matrix to diagonal form:

$$\begin{vmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ -5 & 0 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix} = + \begin{vmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= (-5) \cdot 2 \cdot (-3) \cdot (-1) \cdot 4 = -120.$$

Exercise* 6. Recall that if A is a square matrix, and B is the matrix obtained from A by multiplying a row of A by α , then $\det(B) = \alpha \det(A)$. Suppose now that A is an $n \times n$ matrix. Now, the matrix αA is obtained from A by multiplying each of the n rows of A by α , thus $\det(\alpha A) = \alpha^n \det(A)$.

For the second assertion we proceed by contradiction. Suppose to the contrary that $\det(A+B) = \det(A) + \det(B)$ for all square matrices A and B of the same size. Then, in particular, it would follow that $\det(A+A) = \det(A) + \det(A)$, that is, $\det(2A) = 2\det(A)$. But we just saw that $\det(2A) = 2^n \det(A)$, so $\det(2A) \neq 2\det(A)$ whenever $n > 1$ and $\det(A) \neq 0$. Thus, $\det(A+B)$ cannot equal $\det(A) + \det(B)$ in general.